Recent developments in the theory of universal homogeneous structures

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Motivations

- Classical Fraïssé theory

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- More recent works:

M. DROSTE, R. GÖBEL, *A categorical theorem on universal objects and its application in abelian group theory and computer science*, Proceedings of the International Conference on Algebra, Part 3 (Novosibirsk, 1989), 49–74, Contemp. Math., 131, Part 3, Amer. Math. Soc., 1992.

T. IRWIN, S. SOLECKI, *Projective Fraïssé limits and the pseudo-arc*, Trans. Amer. Math. Soc. **358**, no. 7 (2006) 3077–3096.

W. KUBIŚ, S. SOLECKI, *A proof of uniqueness of the Gurarii space*, Israel J. Math. **195** (2013) 449–456.

I. BEN YAACOV, *Fraïssé limits of metric structures*, Journal of Symbolic Logic **80** (2015), no. 1, 100–115.

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- For every $X, Y \in \mathscr{F}$ there is $V \in \mathscr{F}$ containing both X and Y.
- For every embeddings $f \colon Z \to X$, $g \colon Z \to Y$ with $Z, X, Y \in \mathscr{F}$ there are embeddings $f' \colon X \to W$, $g' \colon Y \to W$ with $f' \circ f = g' \circ g$ and $W \in \mathscr{F}$. (Amalgamation Property)

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- $\mathscr{F}$ has at most countably many isomorphic types.
- $\mathscr{F}$ is hereditary.

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Furthermore, U has the following properties:

- *(Universality) Assume $X = \bigcup_{n \in \omega} X_n$ with $X_n \in \mathscr{F}$, $X_n \subseteq X_{n+1}$, $n \in \omega$. Then X embeds into U.*

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- *(Universality) Assume $X = \bigcup_{n \in \omega} X_n$ with $X_n \in \mathscr{F}$, $X_n \subseteq X_{n+1}$, $n \in \omega$. Then X embeds into U.*
- *(Homogeneity) Every isomorphism between finitely generated substructures of U extends to an automorphism of U.*

The structure U is called the Fraïssé limit of $\mathscr{F}$.

Examples

- $(\mathbb{Q}, \leqslant)$
- The Rado (random) graph
- The universal homogeneous partially ordered set
- The universal homogeneous tournament.

The setup

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(A1) For every $X \in \mathrm{Obj}(\mathfrak{L})$ there exists a sequence $\vec{x} \colon \mathbb{N} \to \mathfrak{S}$ such that $X = \lim \vec{x}$.

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(A2) For every $X = \lim \vec{x} \in \mathrm{Obj}(\mathfrak{L})$, $y \in \mathrm{Obj}(\mathfrak{S})$, for every arrow $f \colon y \to X$ there exists n such that $f = x_n^\infty \circ f'$ for some $f' \in \mathfrak{S}$.

Remark

For every category $\mathfrak{S}$ there exists a category $\sigma\mathfrak{S}$ such that $\langle \mathfrak{S}, \sigma\mathfrak{S} \rangle$ satisfies (A1), (A2).

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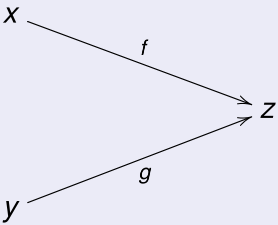
- The objects of $\sigma\mathfrak{S}$ are sequences (i.e., covariant functors) of type $\mathbb{N} \to \mathfrak{S}$.
- The $\sigma\mathfrak{S}$-arrows are natural transformations into subsequences.

Definition

We say that $\mathfrak{S}$ is directed if for every $x, y \in \mathrm{Obj}(\mathfrak{S})$ there exist $z \in \mathrm{Obj}(\mathfrak{S})$ and $\mathfrak{S}$-arrows $f\colon x \to z$, $g\colon y \to z$.



Definition

We say that $\mathfrak{S}$ has the amalgamation property if for every $\mathfrak{S}$-arrows $f\colon z \to x$, $g\colon z \to y$ there exist $\mathfrak{S}$-arrows $f'\colon x \to w$, $g'\colon y \to w$ such that the diagram

$$\begin{array}{ccc} y & \xrightarrow{g'} & w \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle f'} \\ z & \xrightarrow[f]{} & x \end{array}$$

is commutative.

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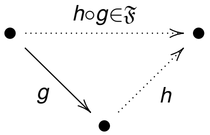
(D2) Given an $\mathfrak{S}$-arrow g with $\mathrm{dom}(g) \in \mathrm{Obj}(\mathfrak{F})$, there exists an $\mathfrak{S}$-arrow h such that $h \circ g \in \mathfrak{F}$.

(D1) $x \dashrightarrow y \in \mathrm{Obj}(\mathfrak{F})$

(D2)



Main definition

Definition

We say that $\mathfrak{S}$ is a Fraïssé category if

- $\mathfrak{S}$ is directed,
- $\mathfrak{S}$ has the amalgamation property,
- $\mathfrak{S}$ is dominated by a countable subcategory.

Theorem (Droste & Göbel 1993, K. 2014)

Assume $\mathfrak{S}$ is a Fraïssé category. Then there exists a unique, up to isomorphism, object $U \in \mathrm{Obj}(\mathfrak{L})$ with the following properties:

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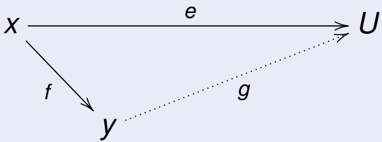
2. *For every $e \colon x \to U$ with $x \in \mathrm{Obj}(\mathfrak{S})$, for every $\mathfrak{S}$-arrow $f \colon x \to y$ there exists an $\mathfrak{L}$-arrow $g \colon y \to U$ such that $e = g \circ f$.*



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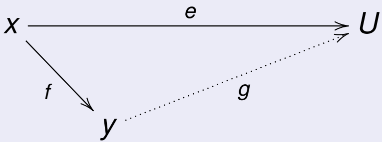


Definition

We call U the Fraïssé limit of $\mathfrak{S}$ and write $U = \mathrm{Flim}(\mathfrak{S})$.

Important features of Fraïssé limits

Theorem (Universality)

Let $U = \mathrm{Flim}(\mathfrak{S})$. *Then for every* $X \in \mathrm{Obj}(\mathfrak{L})$ *there exists an* $\mathfrak{L}$*-arrow* $e\colon X \to U$.

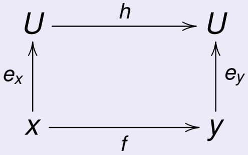
Important features of Fraïssé limits

Theorem (Universality)

Let $U = \mathrm{Flim}(\mathfrak{S})$*. Then for every* $X \in \mathrm{Obj}(\mathfrak{L})$ *there exists an* $\mathfrak{L}$*-arrow* $e \colon X \to U$*.*

Theorem (Homogeneity)

Let $U = \mathrm{Flim}(\mathfrak{S})$*. For every* $\mathfrak{S}$*-arrow* $f \colon x \to y$*, for every* $\mathfrak{L}$*-arrows* $e_x \colon x \to U$*,* $e_y \colon y \to U$ *there exists an automorphism* $h \colon U \to U$ *satisfying* $h \circ e_x = e_y \circ f$*.*



Some examples

Example (Fraïssé)

Let $\mathfrak{S}$ be a category of finitely generated models of a fixed first-order language, $\mathfrak{L}$ a suitable category of countably generated structures. If $\mathfrak{S}$ is hereditary, then $\mathrm{Flim}(\mathfrak{S})$ is the same as the Fraïssé limit in the model-theoretic sense.

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Example (Irwin & Solecki 2006)

Let $\mathfrak{S}$ be a class of finite nonempty structures of some fixed first-order language. Turn it into a category, by saying that f is an arrow from x to y if $f\colon y \to x$ is an epimorphism.
Then $\mathfrak{S}$ is a Fraïssé category $\iff$ $\mathfrak{S}$ is a projective Fraïssé class in the sense of Irwin & Solecki.

Example

Let $\mathfrak{S}$ be the category whose objects are nonempty countable sets and define $\mathfrak{S}(X, Y)$ to be the set of all surjections $f\colon Y \to X$.
Then $\mathfrak{S}$ is a Fraïssé category, yet the set $\mathfrak{S}(\mathbb{N}, \mathbb{N})$ is uncountable.

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Claim

The Fraïssé limit of $\mathfrak{S}$ can be identified as the set of irrational numbers endowed with the natural topology.

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Then $\mathrm{Flim}(\mathfrak{S})$ is the Cantor set 2^ω.

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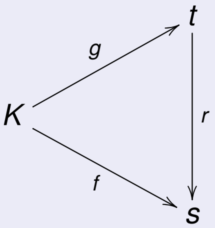
- Given two objects $f\colon K \to s$, $g\colon K \to t$, an $\mathfrak{S}_K$-arrow from f to g is a surjection $r\colon t \to s$ satisfying $r \circ g = f$.



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Corollary (folklore)

Every homeomorphism between closed nowhere dense subsets of 2^ω extends to an auto-homeomorphism of 2^ω.

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So, $\mathbb{U}$ behaves like the Fraïssé limit of $\mathfrak{S}$. How to deal with it?
Note that if $\mathfrak{L}$ is the category of complete separable metric spaces then the pair $\langle \mathfrak{S}, \mathfrak{L} \rangle$ satisfies (A1) but it fails (A2).

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- Eve responds by choosing an $\mathfrak{S}$-arrow $u_1^2 \colon u_1 \to u_2$.
- And so on...

The result is a sequence $\vec{u}$:

$$u_0 \xrightarrow{u_0^1} u_1 \xrightarrow{u_1^2} u_2 \xrightarrow{u_2^3} u_3 \longrightarrow \cdots$$

We say that Odd wins if U is isomorphic to $\lim \vec{u}$. Otherwise Eve wins.

Generic objects

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We say that $U \in \mathrm{Obj}(\mathfrak{L})$ is $\mathfrak{G}$-generic if Odd has a winning strategy in the Banach-Mazur game $\mathrm{BM}(\mathfrak{G}, U)$.

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Proof.

Supposing there are two generic objects and Odd uses his strategy for the first one, Eve can play using Odd's strategy for the second one. □

Theorem

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The converse is false.

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Claim

$\mathfrak{S}$ from the above example has a dominating Fraïssé subcategory.

Question

Assume $\mathfrak{S}$ is countable, $U \in \mathrm{Obj}(\mathfrak{L})$, and Odd has a winning strategy in $\mathrm{BM}(\mathfrak{S}, U)$.
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Fact

Under the assumptions above, $\mathfrak{S}$ is directed.

Proof.

Eve can start the game with an arbitrary $\mathfrak{S}$-object x, showing that there is an $\mathfrak{L}$-arrow $f_x \colon x \to U$.
Taking another $\mathfrak{S}$-object y, we get $f_y \colon y \to U$.
Using (A2), we find m, n such that $f_x = u_m^\infty \circ g_x$ and $f_y = u_n^\infty \circ g_y$ for some $\mathfrak{S}$-arrows g_x, g_y.
Without loss of generality, $n = m$, showing that $\mathfrak{S}$ is directed. □

Metric spaces again

Theorem

Let $\mathfrak{S}$ be the category of finite metric spaces and let $\mathfrak{L}$ be the category of complete separable metric spaces, both with isometric embeddings. Then Odd has a winning strategy in $\mathsf{BM}(\mathfrak{S}, \mathbb{U})$, *where $\mathbb{U}$ is the Urysohn space.*

Banach spaces

Theorem

Let $\mathfrak{S}$ *be the category of finite-dimensional Banach spaces and let* $\mathfrak{L}$ *be the category of separable Banach spaces, both with linear isometric embeddings.*
Then there exists $\mathbb{G} \in \mathrm{Obj}(\mathfrak{L})$ *such that Odd has a winning strategy in* $\mathrm{BM}(\mathfrak{S}, \mathbb{G})$.

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The Banach space $\mathbb{G}$ is known, it is called the Gurariĭ space.
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Remark

The Gurariĭ space $\mathbb{G}$ is not homogeneous, however every linear isometry between its finite-dimensional subspaces can be approximated by bijective linear isometries of $\mathbb{G}$.

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1. $\varrho(f \circ g_1, f \circ g_2) \leqslant \varrho(g_1, g_2)$
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whenever the compositions make sense.

(A2) If $X = \lim \vec{x}$, where $\vec{x}$ is a sequence in $\mathfrak{S}$, then for every $\mathfrak{L}$-arrow $f \colon y \to X$, for every $\varepsilon > 0$ there exist n and an $\mathfrak{S}$-arrow $f' \colon y \to x_n$ such that $\varrho(x_n^\infty \circ f', f) < \varepsilon$.

Domination revisited

Definition

Let $\mathfrak{F}$ be a subcategory of $\mathfrak{S}$. We say that $\mathfrak{F}$ is dominating in $\mathfrak{S}$ if the following conditions are satisfied.

(D1) For every $x \in \mathrm{Obj}(\mathfrak{S})$ there exists an $\mathfrak{S}$-arrow $f \colon x \to y$ such that $y \in \mathrm{Obj}(\mathfrak{F})$.

(D2) Given an $\mathfrak{S}$-arrow g with $\mathrm{dom}(g) \in \mathrm{Obj}(\mathfrak{F})$, for every $\varepsilon > 0$ there exist $h \in \mathfrak{S}$ and $f \in \mathfrak{F}$ such that

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(D1) $x \dashrightarrow y \in \mathrm{Obj}(\mathfrak{F})$

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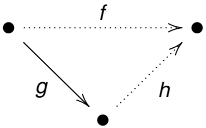

Definition

We say that $\mathfrak{S}$ has the almost amalgamation property if for every $\mathfrak{S}$-arrows $f \colon z \to x$, $g \colon z \to y$, for every $\varepsilon > 0$ there are $\mathfrak{S}$-arrows $f' \colon x \to w$, $g' \colon y \to w$ such that

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Definition

We say that $\mathfrak{S}$ is a Fraïssé category if it is directed, countably dominated and has the almost amalgamation property.

Theorem

Let $\mathfrak{S}$ be a Fraïssé category. There exists a unique, up to isomorphism, $\mathfrak{L}$-object U satisfying

1. *For every $x \in \mathrm{Obj}(\mathfrak{S})$ there exists an $\mathfrak{L}$-arrow $e\colon x \to U$.*
2. *For every $e\colon x \to U$, $f\colon x \to y$, for every $\varepsilon > 0$ there exists $g\colon y \to U$ such that $\varrho(e, g \circ f) < \varepsilon$.*

We say that U is the Fraïssé limit of $\mathfrak{S}$.

Theorem (Universality)

Let U be the Fraïssé limit of $\mathfrak{S}$. Then for every $X \in \mathrm{Obj}(\mathfrak{L})$ there exists an $\mathfrak{L}$-arrow $e\colon X \to U$.

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Theorem (Almost homogeneity)

Let U be the Fraïssé limit of $\mathfrak{S}$. Then for every $\mathfrak{S}$-arrow $f\colon x \to y$, for every $\mathfrak{L}$-arrows $e_x\colon x \to U$, $e_y\colon y \to U$, for every $\varepsilon > 0$ there exists an automorphism $h\colon U \to U$ satisfying

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Remark

The Urysohn space is homogeneous with respect to finite sets, while the Gurariĭ space is not homogeneous with respect to finite-dimensional spaces.

Example

Let $\mathfrak{S}$ be the category whose objects are closed intervals $[0, n]$ ($n \in \mathbb{N}$) and arrows are non-expansive surjections. More precisely, $f \in \mathfrak{S}([0, n], [0, m])$ iff f is a non-expansive surjection from $[0, m]$ onto $[0, n]$.

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Fact

$\mathfrak{S}$ is a Fraïssé category, although it fails the amalgamation property.

$\mathfrak{L}$ is the category of all nonempty *chainable continua* (a continuum = a compact metrizable connected space).
The Fraïssé limit of $\mathfrak{S}$ is the *pseudo-arc*.

Bad news

Fact

The category of finite metric spaces with isometric embeddings is not countably dominated.

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Fact

The category of finite-dimensional Banach spaces with linear isometric embeddings is not countably dominated.

Proposition

A separable Banach space G is linearly isometric to the Gurariĭ space if and only if

(G) *For every finite-dimensional spaces* $X \subseteq Y$*, for every linear isometric embedding* $e\colon X \to G$*, for every* $\varepsilon > 0$ *there exists an* ε*-isometric embedding* $f\colon Y \to G$ *such that* $\|f \restriction X - e\| < \varepsilon$*.*

Measured categories

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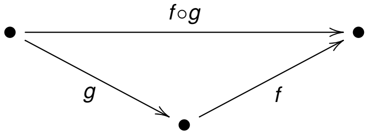
(M3) $\mu(g) \leqslant \mu(f \circ g) + \mu(f)$ whenever $f \circ g$ is defined.

A pair $\langle \mathfrak{K}, \mu \rangle$ will be called a measured category.



Example

Let $\mathfrak{K}$ be the category of metric spaces with non-expansive mappings. Then

$$\mu(f) = \log Lip(f^{-1})$$

defines a measure on $\mathfrak{K}$.

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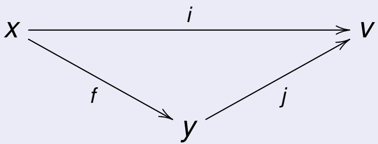
Example

Let $\mathfrak{K} = \langle X, X \times X \rangle$ be a quasi-ordered set, treated as a category such that $\mathfrak{K}(x, y) = \{\langle x, y \rangle\}$ for every $x, y \in X$. Then a measure on $\langle X, \leqslant \rangle$ is a pseudo-metric (we allow 0 for distinct points).

We assume that $\mathfrak{S}$ is a measured category enriched over metric spaces.

A new axiom

For every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $f\colon x \to y$ satisfies $\mu(f) < \delta$ then there exist $i\colon x \to v$, $j\colon y \to v$ such that $\mu(i) = \mu(j) = 0$ and $\varrho(i, j \circ f) < \varepsilon$.



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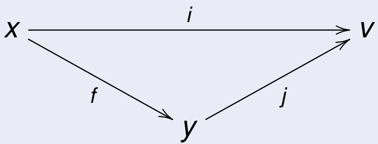

Proposition

The category of finite-dimensional Banach spaces satisfies this axiom (with $\delta = \varepsilon$).

After adapting the other assumptions and axioms, we obtain the final notion of a Fraïssé category.

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Theorem

The Urysohn space is the Fraïssé limit of the category of finite metric spaces.

Theorem

The Gurariĭ space is the Fraïssé limit of the category of finite-dimensional Banach spaces.

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